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On the zeros of $J_n(z) \pm iJ_{n+1}(z)$ and $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$

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Abstract

The zeros of $J_n(z) \pm i J_{n+1}(z)$ and $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$ play an important role in certain physical applications. At the origin these functions have a zero of multiplicity n (if $n \geqslant 1$) and $2n+2$, respectively. We prove that all the zeros that lie in $\mathbb{C}_0$ are simple. ZEBEC (Kravanja et al., Comput. Phys. Commun. 113(2–3) (1998) 220–238) is a reliable software package for calculating zeros of Bessel functions of the first, the second, or the third kind, or their first derivatives. It can be easily extended to calculate zeros of any analytic function, provided that the zeros are known to be simple. Thus, ZEBEC is the package of choice to calculate the zeros of $J_n(z) \pm i J_{n+1}(z)$ or $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$. We tabulate the first 30 zeros of $J_5(z) - i J_6(z)$ and $J_{10}(z) - i J_{11}(z)$ that lie in the fourth quadrant as computed by ZEBEC. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We focus on the complex zeros of the entire functions

$$z \mapsto J_n(z) \pm i J_{n+1}(z) \tag{1}$$

and

$$z \mapsto [J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z), \tag{2}$$

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where $n \in \mathbb{N}$ and $J_n(z)$ denotes the Bessel function of the first kind of order n,

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{2k}, \quad |z| < \infty. \tag{3}$$

Let $F_n(z) := J_n(z) - \mathrm{i}\,J_{n+1}(z)$ and $G_n(z) := [J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$.

The zeros of $F_n(z)$ and $G_n(z)$ play an important role in certain physical applications. The zeros of $F_0(z)$ are of interest to the specialist in the problem of water wave run-up on a sloping beach, (cf. [8]). The equation $F_n(z) = 0$ arises in problems of wave reflection from composite beaches, i.e., beaches with multiple slopes [10]. MacDonald [6] used the zeros of $G_0(z)$ to plot representative streamlines for the steady motion of a viscous fluid in a long tube, of constant radius, which rotates about its axis (the $\hat{z}$-axis) with an angular velocity that changes discontinuously at $\hat{z} = 0$ from one constant value to another of the same sign.

MacDonald derived asymptotic formulae for the zeros of $F_0(z)$, $F_n(z)$ and $G_n(z)$ in [5,6] and [7], respectively. These formulae permit to locate zeros of large modulus. MacDonald observed that the accuracy of the asymptotic formulae for the zeros of $F_n(z)$ deteriorates as n increases. Also, asymptotic formulae may be inadequate for smaller zeros. These have to be computed by some numerical procedure. To obtain the smaller zeros of $F_n(z)$, MacDonald [7] truncated the ascending series of $F_n(z)$ and calculated the zeros of the truncated series via Newton's method.

In this paper we suggest to use the software package ZEBEC [4] to compute zeros of (1) or (2). ZEBEC is a reliable package for calculating zeros of Bessel functions. More specifically, given a rectangular region in the complex plane, ZEBEC is able to compute all the zeros of $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ or their first derivatives, where $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\nu \in \mathbb{R}$, that lie inside this region. The package can be easily extended to calculate zeros of any analytic function, provided that the zeros are known to be simple. We will show that all the zeros of (1) and (2) that lie in $\mathbb{C}_0$ are simple. Hence ZEBEC is the package of choice to calculate these zeros. [1]

This paper is organized as follows. In Section 2, we list some well-known properties of the Bessel function $J_n(z)$ that we need in the sequel. In Sections 3 and 4, we summarize relevant properties of the zeros of (1) and (2), respectively. At $z = 0$ these functions have a zero of multiplicity n (if $n \geqslant 1$) and $2n + 2$, respectively. We prove that all the zeros that lie in $\mathbb{C}_0$ are simple. In Section 5, we examine how ZEBEC can be used to calculate zeros of (1) or (2). We tabulate the first 30 zeros of $J_5(z) - \mathrm{i}\,J_6(z)$ and $J_{10}(z) - \mathrm{i}\,J_{11}(z)$ that lie in the fourth quadrant as computed by ZEBEC.

2. Properties of $J_n(z)$

We will need the following properties of $J_n(z)$.

- Symmetry (this follows immediately from (3)):

$$J_n(\bar{z}) = \overline{J_n(z)}, \quad J_n(-z) = (-)^n J_n(z). \tag{4}$$

[1] The package is part of the CPC Program Library. For more information on how to obtain ZEBEC, we refer the reader to the URL http://www.cpc.cs.qub.ac.uk/cpc/cgi-bin/list_summary.pl/?CatNumber=ADIO.

- Differentiation formulae [2, Section 7.2.8]:

$$J_n'(z) = \frac{n}{z} J_n(z) - J_{n+1}(z), \tag{5}$$

$$J_{n+1}'(z) = J_n(z) - \frac{n+1}{z} J_{n+1}(z). \tag{6}$$

- Recurrence relation [2, Section 7.2.8]:

$$J_{n+2}(z) = \frac{2(n+1)}{z} J_{n+1}(z) - J_n(z). \tag{7}$$

- The Poisson integral representation [2, formula (7), p. 81]:

$$\Gamma\left(n + \frac{1}{2}\right) J_n(z) = \frac{1}{\sqrt{\pi}} \left(\frac{z}{2}\right)^n \int_{-1}^{1} e^{izt}(1-t^2)^{n-(1/2)}\, dt. \tag{8}$$

- Every zero of $J_n(z)$ is simple, the only possible exception being the origin [2, Section 7.2.9]. In view of (5) this can also be expressed as follows:

$$J_n(z) = J_{n+1}(z) = 0 \Rightarrow z = 0, \quad n > 0. \tag{9}$$

- The inequality (cf. [9])

$$[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z) > \frac{[J_{n+1}(z)]^2}{n+2}, \quad z > 0. \tag{10}$$

3. The zeros of $J_n(z) \pm iJ_{n+1}(z)$

The symmetry properties (4) imply that

$$J_n(z) + i J_{n+1}(z) = \overline{F_n(\bar{z})} = (-)^n F_n(-z),$$

where $F_n(z) = J_n(z) - i J_{n+1}(z)$. Hence, the zeros of $J_n(z) + i J_{n+1}(z)$ are the reflections of the zeros of $F_n(z)$ about the real axis and these zeros are symmetric with respect to the imaginary axis. Therefore, we will restrict our attention to the zeros of the function $F_n(z)$ in the right half-plane.

If $n \geqslant 1$, then $J_n(z)$ has a zero of multiplicity n at the origin, whereas $J_{n+1}(z)$ has a zero of multiplicity $n+1$ at the origin. Therefore, when $n \geqslant 1$, the function $F_n(z)$ has a zero of multiplicity n at the origin. However, it has no other real zeros.

Theorem 1. *Except for $z = 0$, the function $F_n(z)$ has no zeros on the real axis.*

Proof. Suppose that $z^\star \in \mathbb{R}_0$ is such that

$$J_n(z^\star) - i J_{n+1}(z^\star) = 0.$$

As $J_n(z)$ and $J_{n+1}(z)$ take real values on the real axis, it follows that $J_n(z^\star) = J_{n+1}(z^\star) = 0$. However, this is impossible by (9). □

Except for $z = 0$, the function $F_n(z)$ has no zeros on the imaginary axis. This is a corollary of the following theorem.

Theorem 2. *The following integral representation holds*:

$$J_n(z) - \mathrm{i}\,J_{n+1}(z) = \frac{1}{\sqrt{\pi}\,\Gamma(n+(1/2))}\left(\frac{z}{2}\right)^n \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1+t)^{n-(1/2)}(1-t)^{n+(1/2)}\,\mathrm{d}t.$$

Proof. By replacing n by $n+1$ in (8), we obtain

$$\Gamma\left(n+\frac{3}{2}\right) J_{n+1}(z) = \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{n+1} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{n+(1/2)}\,\mathrm{d}t.$$

We integrate the integral on the right-hand side by parts and divide the equation by $n+(1/2)$. This gives

$$\Gamma\left(n+\frac{1}{2}\right) J_{n+1}(z) = \frac{1}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{n+1} \int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{i}zt}}{iz}(1-t^2)^{n-(1/2)}2t\,\mathrm{d}t. \tag{11}$$

By adding (8) and (11) one obtains the given integral representation for $J_n(z) - \mathrm{i}\,J_{n+1}(z)$. □

Note: The functions $J_n(z) \pm \mathrm{i}\,J_{n+1}(z)$ also play a role in asymptotics of orthogonal polynomials on the unit circle [1,12]. The integral representation given in the previous theorem can also be found in [12].

Corollary 3. *Except for* $z=0$, *the functions* $F_n(z)$ *have no zeros on the imaginary axis.*

Proof. This follows immediately from the fact that the integrand (and thus also the integral) in the representation given in the previous theorem is positive if z lies on the imaginary axis. □

Theorem 4. *All the zeros of* $F_n(z)$ *lie in the lower half-plane.*

Proof. See [10]. □

We already know that $F_n(z)$ has a zero of multiplicity n at the origin if $n \geqslant 1$. The following theorem tells us that all the other zeros are simple.

Theorem 5. *The zeros of* $F_n(z)$ *that lie in* $\mathbb{C}_0$ *are simple.*

Proof. Suppose that $z^\star \in \mathbb{C}_0$ is a multiple zero of $F_n(z) = J_n(z) - \mathrm{i}\,J_{n+1}(z)$. Then

$$F_n(z^\star) = J_n(z^\star) - \mathrm{i}\,J_{n+1}(z^\star) = 0 \tag{12}$$

and

$$F_n'(z^\star) = J_n'(z^\star) - \mathrm{i}\,J_{n+1}'(z^\star) = 0.$$

By using the differentiation formulae (5) and (6), we obtain that

$$F_n'(z^\star) = \left(\frac{n}{z^\star} - \mathrm{i}\right) J_n(z^\star) - \left(1 - \mathrm{i}\,\frac{n+1}{z^\star}\right) J_{n+1}(z^\star) = 0. \tag{13}$$

Eqs. (12) and (13) represent a homogeneous linear system in $J_n(z^\star)$ and $J_{n+1}(z^\star)$. Its determinant is equal to $\mathrm{i}(2n+1)/z^\star \neq 0$. It follows that $J_n(z^\star) = J_{n+1}(z^\star) = 0$, which is in contradiction with (9). This proves that all the zeros of $F_n(z)$ that lie in $\mathbb{C}_0$ are simple. □

4. The zeros of $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$

If $z^\star$ is a zero of $G_n(z) = [J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$, then (4) immediately implies that the same holds for $\overline{z^\star}$ and $-z^\star$. In other words, the zeros of $G_n(z)$ are symmetric with respect to the real axis and about the origin. Thus, we may restrict our attention to one quadrant in the complex plane, for example the first quadrant.

If $n \geqslant 1$, then $J_n(z)$ has a zero of multiplicity n at the origin. The Bessel functions $J_{n+1}(z)$ and $J_{n+2}(z)$ have a zero of multiplicity $n+1$, $n+2$, respectively, at the origin. Therefore, if $n \geqslant 1$, then $G_n(z) = [J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$ has a zero of multiplicity $\geqslant 2n+2$ at the origin. Using Taylor series (3) one can easily verify that the multiplicity is in fact equal to $2n+2$. This holds also in the case $n = 0$. Inequality (10) implies that no zero exists on the positive real axis, and, by symmety, neither on the negative real axis. In Corollary 7, we will show that no zeros exist on the imaginary axis, except the origin.

Theorem 6. *$G_n'(z) = (2/z)J_n(z)J_{n+2}(z)$ for all $z \neq 0$.*

Proof. Using the differentiation formulae (5) and (6), we have

$$\begin{aligned} G_n'(z) &= 2J_{n+1}(z)J_{n+1}'(z) - J_n(z)J_{n+2}'(z) - J_n'(z)J_{n+2}(z) \\ &= 2J_{n+1}(z)\left(J_n(z) - \frac{n+1}{z}J_{n+1}(z)\right) - J_n(z)\left(J_{n+1}(z) - \frac{n+2}{z}J_{n+2}(z)\right) \\ &\quad - J_{n+2}(z)\left(\frac{n}{z}J_n(z) - J_{n+1}(z)\right) \\ &= \frac{2}{z}J_n(z)J_{n+2}(z) + J_{n+1}(z)\left(J_{n+2}(z) + J_n(z) - \frac{2(n+1)}{z}J_{n+1}(z)\right). \end{aligned}$$

By the recurrence relation (7), the last term vanishes and the proof is complete. □

Corollary 7. *Except for $z = 0$, the function $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$ has no zeros on the imaginary axis.*

Proof. Define $f_n(\zeta) : \mathbb{R} \to \mathbb{R} : \zeta \mapsto G_n(\mathrm{i}\zeta)$. Suppose this function has a real zero besides $\zeta = 0$. Then by Rolle's theorem, its derivative vanishes in some real point between this zero and $\zeta = 0$. However, this is impossible since $f_n'(\zeta)$ vanishes only at the origin. Indeed, using Theorem 6 and the Taylor expansion (3), we have

$$\begin{aligned} f_n'(\zeta) &= \mathrm{i}G_n'(\mathrm{i}\zeta) = \frac{2}{\zeta}J_n(\mathrm{i}\zeta)J_{n+2}(\mathrm{i}\zeta) \\ &= (-)^{n+1}\zeta^{2n+1}\left(\sum_{k=0}^{\infty}\frac{1}{k!(n+k)!}\left(\frac{\zeta}{2}\right)^{2k}\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!(n+2+k)!}\left(\frac{\zeta}{2}\right)^{2k}\right). \end{aligned}$$

The infinite sums in this expression are positive for real ζ, since each of their terms is positive. Hence, $f_n'(\zeta)$ vanishes only at the origin. □

Corollary 8. *The zeros of* $[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$ *that lie in* $\mathbb{C}_0$ *are simple.*

Proof. Suppose that $z^\star \in \mathbb{C}_0$ is a multiple zero of $G_n(z)$, then

$$G_n(z^\star) = [J_{n+1}(z^\star)]^2 - J_n(z^\star)J_{n+2}(z^\star) = 0,$$
$$G_n'(z^\star) = \frac{2}{z^\star} J_n(z^\star)J_{n+2}(z^\star) = 0.$$

Solving this system in $J_n(z^\star)$, $J_{n+1}(z^\star)$ and $J_{n+2}(z^\star)$, we have either

$$J_n(z^\star) = J_{n+1}(z^\star) = 0$$

or

$$J_{n+2}(z^\star) = J_{n+1}(z^\star) = 0.$$

However, both cases are in contradiction with (9). □

5. Numerical results

Kravanja et al. [4] wrote a reliable software package, called ZEBEC, for computing zeros of Bessel functions. More specifically, the package can calculate all the zeros of $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$, or their first derivatives, where $z \in \mathbb{C} \setminus (-\infty, 0]$ and $\nu \in \mathbb{R}$, that lie inside a given rectangle whose edges are parallel to the coordinate axes. The Bessel functions are evaluated via the package BESSCC written by Thompson and Barnett [11].

ZEBEC starts by computing the total number of zeros N that lie inside the given rectangular box. This is done by calculating the logarithmic residue integral [3]

$$N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z, \tag{14}$$

where $f(z)$ denotes one of the four Bessel functions that we have just mentioned or its first derivative, and γ is the boundary of the given box. If the box contains only one zero, then this zero is calculated, up to a requested accuracy, via the package CHABIS [13], which implements a generalized method of bisection called characteristic bisection. Else, the box is subdivided into two equal boxes by halving its longest edge, and the whole process is repeated: the value of (14) is calculated for one of these boxes, etc. For more details, we refer to Kravanja et al. [4].

ZEBEC was written for the Bessel functions $J_\nu(z)$, $Y_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ and their first derivatives, but it can be easily extended to calculate zeros of any analytic function $f(z)$ whose zeros are known to be simple, provided that a Fortran-77 routine exists to evaluate the logarithmic derivative $f'(z)/f(z)$. In Sections 3 and 4 we have shown that all the zeros of $F_n(z) = J_n(z) - \mathrm{i}J_{n+1}(z)$ and $G_n(z) = [J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)$ that lie in $\mathbb{C}_0$ are simple. The zeros of these functions can therefore be calculated via ZEBEC. As

$$\frac{F_n'(z)}{F_n(z)} = \frac{J_n'(z) - \mathrm{i}J_{n+1}'(z)}{J_n(z) - \mathrm{i}J_{n+1}(z)} \quad \text{and} \quad \frac{G_n'(z)}{G_n(z)} = \frac{(2/z)J_n(z)J_{n+2}(z)}{[J_{n+1}(z)]^2 - J_n(z)J_{n+2}(z)},$$

it seems that one has to evaluate $J_n(z)$, $J_n'(z)$, $J_{n+1}(z)$, $J_{n+1}'(z)$ and $J_n(z)$, $J_{n+1}(z)$, $J_{n+2}(z)$ to compute $F_n'(z)/F_n(z)$ and $G_n'(z)/G_n(z)$, respectively. However, relations (5)–(7) imply that it is sufficient to compute $J_n(z)$ and $J_{n+1}(z)$.

Table 1
Approximations for the first 30 zeros of $J_5(z) - \mathrm{i}J_6(z)$ that lie in the fourth quadrant

9.33311962903697−i0.69444271743387
12.94510369182598−i0.82936671338824
16.33461779337308−i0.93080334205489
19.63429282581447−i1.01367717572733
22.88679602216657−i1.08418924023245
26.11097880771598−i1.14572183934914
29.31666723089333−i1.20038033303009
32.50954914896803−i1.24958398094783
35.69316831642123−i1.29434274725774
38.86985731122473−i1.33540436964299
42.04121867038544−i1.37333958320706
45.20839196877325−i1.40859475982058
48.37221090307405−i1.44152605770887
51.53330011302131−i1.47242245993473
54.69213722132923−i1.50152182265512
57.84909391098780−i1.52902235758964
61.00446389725773−i1.55509103690239
64.15848244476649−i1.57986986761685
67.31134027894330−i1.60348065661451
70.46319369060231−i1.62602868463876
73.61417199987465−i1.64760557747193
76.76438315327531−i1.66829157701086
79.91391797841353−i1.68815735731665
83.06285345843724−i1.70726549130409
86.21125528062697−i1.72567164594395
89.35917984058424−i1.74342556430824
92.50667583333037−i1.76057187870224
95.65378552754606−i1.77715078859128
98.80054579435346−i1.79319862955800
101.94698894421000−i1.80874835363552

We have calculated the first 30 zeros of $J_5(z) - \mathrm{i}J_6(z)$ that lie in the fourth quadrant. The results are shown in Table 1. The reader may compare these with the values given by MacDonald [7, Table 7, p. 633]. ZEBEC uses the following stopping criterion. Termination occurs if CHABIS estimates that the modulus of the function value at an approximate zero is at most ε, or if the diameter of the box that represents an approximate zero is less than 4ε. We have set $\varepsilon = 10^{-13}$.

We have also calculated the first 30 zeros of $J_{10}(z) - \mathrm{i}J_{11}(z)$ that lie in the fourth quadrant, a case not previously discussed in the literature. The results are shown in Table 2.

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Table 2
Approximations for the first 30 zeros of $J_{10}(z) - \mathrm{i}J_{11}(z)$ that lie in the fourth quadrant

15.021420363907497−i0.627782803840525
19.014059158522549−i0.719949130085605
22.651275391307930−i0.792047623539251
26.131693624487980−i0.852950202825426
29.523928376877350−i0.906238608297139
32.860183981208870−i0.953854097521185
36.158144350930797−i0.997013164648797
39.428520940422771−i1.036547328483510
42.678258456257822−i1.073058771672258
45.912091819531234−i1.107001933931306
49.133377543623780−i1.138730521748051
52.344570350377673−i1.168526548456019
55.547511990352746−i1.196619259617420
58.743614254745424−i1.223197999513516
61.933979289999812−i1.248421263079944
65.119481179021648−i1.272423246750445
68.300822738116750−i1.295318703026888
71.478575971804048−i1.317206611466725
74.653211468104786−i1.338173003350145
77.825120137022495−i1.358293168470762
80.994629540319522−i1.377633402306960
84.162016331631605−i1.396252405874709
87.327515854095608−i1.414202419188905
90.491329630364461−i1.431530147898935
93.653631269291537−i1.448277527372106
96.814571168806069−i1.464482357708114
99.974280293350148−i1.480178835365255
103.132873232791553−i1.495398001100066
106.290450698100656−i1.510168119779518
109.447101571811260−i1.524515004216060

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Further reading

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C.E. Synolakis, On the roots of $f(z) = J_0(z) - \mathrm{i}J_1(z)$, Quart. Appl. Math. 46 (1) (1988) 105–107.