It follows that no matter what pair (f, g) we take from $\mathscr{B}$, if we put $d_1 = f, d_2 = g$ and define d_j for $3 \le j \le sn + 2s + 2$ by

$$d_j = e_{j-2} d_{j-1} + d_{j-2},$$

then there is a value j with

$$d_{j+i} \equiv b_i (\text{mod } m), \quad \text{for} \quad 1 \le i \le n.$$

By Theorem 2 almost all α have infinitely many sets of consecutive partial quotients of the form $\{e_1, \dots, e_{sn+2s}\}$, and this completes the proof.

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Multidimensional Analytic Deflation

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1. Introduction. Consider an analytic function $f: \mathbb{C} \to \mathbb{C}$ and suppose that a zero $z^* \in \mathbb{C}$ of f is known, so $f(z^*) = 0$. How can one get rid of this zero? A standard way is to divide $f(z)$ by $z - z^*$. The function

$$\mathscr{B}(z) := \mathscr{B}(z; f, z^*) := \begin{cases} \dfrac{f(z)}{z - z^*}, & \text{if } z \neq z^* \\ f'(z^*), & \text{if } z = z^* \end{cases}$$

has the following well-known properties:

- $\mathscr{B}$ is analytic.
- The zeros of $\mathscr{B}$ are the zeros of f, except for z^*.
- If z^* is a zero of multiplicity μ of f, then z^* is a zero of multiplicity $\mu - 1$ of $\mathscr{B}$. In particular, if z^* is a simple zero of f, $f'(z^*) \neq 0$, then z^* is not a zero of $\mathscr{B}$.

The process of constructing $\mathscr{B}$ from f is called *analytic deflation.* It is a well-known and useful tool in the problem of computing all the zeros of f. The term 'analytic' stresses the fact that $\mathscr{B}$ is again an analytic function. One starts with an analytic function f and one of its zeros z^* and ends up with a function that is again analytic and has the same zeros as f, except for z^*.

What happens in the multidimensional case? What can be said if f is a vector of analytic functions of several variables? We show that the role of the function $\mathscr{B}$ can be played by a certain determinant that is sometimes called a *multivariate Bezoutian.*

2. Multivariate Bezoutians. Let $n \geq 2$ be an integer and let $f = f(z): \mathbb{C}^n \to \mathbb{C}^n$ be an analytic mapping,where $z = (z_1, \ldots, z_n)$ and $f = (f_1, \ldots, f_n)$. Suppose that an isolated zero $z^* = (z_1^*, \ldots, z_n^*) \in \mathbb{C}^n$ of f is known, $f(z^*) = 0$.

One can insert the components of z^* into z step by step as follows. Define

$$z^{\langle k \rangle} := (z_1^*, \ldots, z_k^*, z_{k+1}, \ldots, z_n)$$

for $k = 1, \ldots, n-1$. Let $z^{\langle 0 \rangle} := z$ and $z^{\langle n \rangle} := z^*$.

Define the kth *discrete differentiation* of the jth component f_j as

$$\theta_k(f_j) := \frac{f_j(z^{\langle k-1 \rangle}) - f_j(z^{\langle k \rangle})}{z_k - z_k^*}, \quad z_k \neq z_k^*$$

for $k, j = 1, \ldots, n$. If $z_k = z_k^*$, then $\theta_k(f_j)$ is defined by taking the corresponding limit. Now define the function $\mathscr{B}: \mathbb{C}^n \to \mathbb{C}$ as follows: if all the components of z are different from the corresponding components of z^*, then

$$\mathscr{B}(z) := \mathscr{B}(z; f, z^*) := \det \begin{bmatrix} \theta_1(f_1) & \cdots & \theta_n(f_1) \\ \vdots & & \vdots \\ \theta_1(f_n) & \cdots & \theta_n(f_n) \end{bmatrix}. \tag{1}$$

If some components of z are equal to the corresponding components of z^*, then $\mathscr{B}(z)$ is defined by taking the limit. Note that $\mathscr{B}$ is a complex function of n complex variables, not a mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. It looks like some sort of 'discrete Jacobian' of f relative to the point z^*. The following result is easy to prove.

Theorem 1. *The function $\mathscr{B}: \mathbb{C}^n \to \mathbb{C}$ is analytic.*

What does $\mathscr{B}$ do at the zeros of f?

Theorem 2. *Let $z \in \mathbb{C}^n$, $z \neq z^*$. If $f(z) = 0$, then $\mathscr{B}(z) = 0$.*

Proof: Suppose first that z has no component in common with z^*. Then

$$\mathscr{B}(z) = \frac{1}{(z_1 - z_1^*) \cdots (z_n - z_n^*)}$$

$$\times \det \begin{bmatrix} f_1(z) - f_1(z^{\langle 1 \rangle}) & f_1(z^{\langle 1 \rangle}) - f_1(z^{\langle 2 \rangle}) & \cdots & f_1(z^{\langle n-1 \rangle}) - f_1(z^*) \\ \vdots & \vdots & & \vdots \\ f_n(z) - f_n(z^{\langle 1 \rangle}) & f_n(z^{\langle 1 \rangle}) - f_n(z^{\langle 2 \rangle}) & \cdots & f_n(z^{\langle n-1 \rangle}) - f_n(z^*) \end{bmatrix}.$$

For $k = 1, \ldots, n-1$, we replace the kth column in the determinant in the right-hand side by the sum of columns $k, k+1, \ldots, n$. It follows that

$$\mathscr{B}(z) = \frac{1}{(z_1 - z_1^*) \cdots (z_n - z_n^*)} \det \begin{bmatrix} f_1(z) & f_1(z^{\langle 1 \rangle}) & \cdots & f_1(z^{\langle n-1 \rangle}) \\ \vdots & \vdots & & \vdots \\ f_n(z) & f_n(z^{\langle 1 \rangle}) & \cdots & f_n(z^{\langle n-1 \rangle}) \end{bmatrix}.$$

Thus if $f(z) = 0$, then indeed $\mathscr{B}(z) = 0$.

What happens if some components of z are equal to the corresponding components of z^*? As $z \neq z^*$, it is impossible that all the components of z are equal to the corresponding components of z^*. Let us assume that only one component of z is equal to the corresponding component of z^*, for example the first component, $z_1 = z_1^*$. Then

$$\mathscr{B}(z) = \frac{1}{(z_2 - z_2^*) \cdots (z_n - z_n^*)} \times$$

$$\det \begin{bmatrix} \lim\limits_{z_1 \to z_1^*} \dfrac{f_1(z) - f_1(z^{\langle 1 \rangle})}{z_1 - z_1^*} & f_1(z^{\langle 1 \rangle}) - f_1(z^{\langle 2 \rangle}) & \cdots & f_1(z^{\langle n-1 \rangle}) - f_1(z^*) \\ \vdots & \vdots & & \vdots \\ \lim\limits_{z_1 \to z_1^*} \dfrac{f_n(z) - f_n(z^{\langle 1 \rangle})}{z_1 - z_1^*} & f_n(z^{\langle 1 \rangle}) - f_n(z^{\langle 2 \rangle}) & \cdots & f_n(z^{\langle n-1 \rangle}) - f_n(z^*) \end{bmatrix}.$$

By replacing columns in the determinant in the right-hand side by sums of columns as before and by taking into account that $z = z^{\langle 1 \rangle}$, it follows that

$$\mathscr{B}(z) = \frac{1}{(z_2 - z_2^*) \cdots (z_n - z_n^*)}$$

$$\times \det \begin{bmatrix} \lim\limits_{z_1 \to z_1^*} \dfrac{f_1(z) - f_1(z^{\langle 1 \rangle})}{z_1 - z_1^*} & f_1(z) & f_1(z^{\langle 2 \rangle}) & \cdots & f_1(z^{\langle n-1 \rangle}) \\ \vdots & \vdots & \vdots & & \vdots \\ \lim\limits_{z_1 \to z_1^*} \dfrac{f_n(z) - f_n(z^{\langle 1 \rangle})}{z_1 - z_1^*} & f_n(z) & f_n(z^{\langle 2 \rangle}) & \cdots & f_n(z^{\langle n-1 \rangle}) \end{bmatrix}.$$

Thus if $f(z) = 0$, then indeed $\mathscr{B}(z) = 0$. The other cases ($z_2 = z_2^*, \ldots, z_n = z_n^*$ and also the case in which several corresponding components are equal) can be handled in an analogous way. ■

An interesting result, isn't it. All the solutions to the system of analytic equations $f(z) = 0$, except for $z = z^*$, about which the theorem doesn't say anything, satisfy the equation $\mathscr{B}(z) = 0$. And what happens at z^*? One can easily prove the following.

Theorem 3. *Let $f'(z) \in \mathbb{C}^{n \times n}$ denote the Jacobian matrix of f at the point $z \in \mathbb{C}^n$. Then $\mathscr{B}(z^*) = \det f'(z^*)$. Thus, if z^* is a simple zero of the mapping f, $\det f'(z^*) \neq 0$, then $\mathscr{B}(z^*) \neq 0$.*

Therefore, if z^* is a simple zero of f, we have

$$\begin{cases} f(z) = 0 \\ \mathscr{B}(z) = 0 \end{cases} \Rightarrow f(z) = 0 \quad \text{and } z \neq z^*. \tag{2}$$

The overdetermined system ($n + 1$ equations in n variables) in the left-hand side represents an analytic deflation of the system $f(z) = 0$ relative to z^*.

3. An example. The multivariate Bezoutian (1) is defined for arbitrary analytic mappings, but it is especially useful for polynomial mappings. Consider the system of polynomial equations

$$\begin{cases} (x-1)(x+y+z-1) & = & 0 \\ y(x+y+z-3) & = & 0 \\ (z+1)(x+y^2-z^2-1) & = & 0. \end{cases}$$

This system has five finite solutions: $(1,0,-1)$, $(1,0,0)$, $(1,3,-1)$, $(1,1,1)$, and $(2,0,-1)$. The Bezoutian with respect to $(1,0,-1)$ is

$$\mathscr{B}(x,y,z) =$$

$$\det\begin{bmatrix} \frac{(x-1)(x+y+z-1)-0}{x-1} & \frac{0-0}{y} & \frac{0-0}{z+1} \\ \frac{y(x+y+z-3)-y(y+z-2)}{x-1} & \frac{y(y+z-2)-0}{y} & \frac{0-0}{z+1} \\ \frac{(z+1)(x+y^2-z^2-1)-(z+1)(y^2-z^2)}{x-1} & \frac{(z+1)(y^2-z^2)-(z+1)(-z^2)}{y} & \frac{(z+1)(-z^2)-0}{z+1} \end{bmatrix}$$

$$= \det\begin{bmatrix} x+y+z-1 & 0 & 0 \\ y & y+z-2 & 0 \\ z+1 & y(z+1) & -z^2 \end{bmatrix}$$

$$= (x+y+z-1)(y+z-2)(-z^2)$$

and indeed

$$\mathscr{B}(1,0,0) = \mathscr{B}(1,3,-1) = \mathscr{B}(1,1,1) = \mathscr{B}(2,0,-1) = 0,$$

whereas $\mathscr{B}(1,0,-1) \neq 0$.

4. An open problem. Is it possible to define a multidimensional Bezoutian that is independent of the choice of basis? We don't know, but want to give the reader an impression of the problems involved in generalizing our definition of $\mathscr{B}(z)$. Consider the linear transformation $\tilde{z} := zM$, where $z = (z_1, \ldots, z_n)$ as before, $\tilde{z} = (\tilde{z}_1, \ldots, \tilde{z}_n)$, and $M \in \mathbb{C}^{n\times n}$ is a nonsingular matrix. The mapping f is transformed as $\tilde{f}(\tilde{z}) := f(\tilde{z}M^{-1})$. Let $\tilde{z}^* := z^*M$ be the transformed zero. Then one can define a Bezoutian $\tilde{\mathscr{B}}(\tilde{z}) = \mathscr{B}(\tilde{z}; \tilde{f}, \tilde{z}^*)$ in the new coordinates $\tilde{z}$. Is there a connection between $\mathscr{B}(z)$ and $\tilde{\mathscr{B}}(zM)$? If M is a diagonal matrix, then $\tilde{\mathscr{B}}(zM) = [\det M]^{-1}\mathscr{B}(z)$. For general M, experiments with polynomial mappings f have led us to conjecture that in this case

$$\tilde{\mathscr{B}}(zM) = [\det M]^{-1}\mathscr{B}(z) \bmod (\mathrm{f}).$$

A major complication in tackling this generalization of $\mathscr{B}(z)$ is the fact that $\tilde{z} = zM$ and $\tilde{z}^* = z^*M$ do *not* imply that $\tilde{z}^{\langle k\rangle} = z^{\langle k\rangle}M$.

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Irrationality of The Square Root of Two —A Geometric Proof

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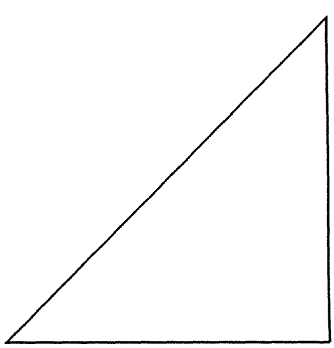
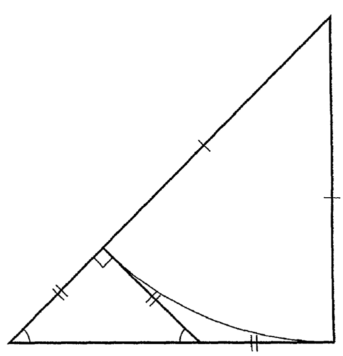
This note presents a remarkably simple proof of the irrationality of $\sqrt{2}$ that is a variation of the classical Greek geometric proof.

By the Pythagorean theorem, an isosceles right triangle of edge-length 1 has hypotenuse of length $\sqrt{2}$. If $\sqrt{2}$ is rational, some positive integer multiple of this triangle must have three sides with integer lengths, and hence there must be a *smallest* isosceles right triangle with this property. But inside any isosceles right triangle whose three sides have integer lengths we can always construct a smaller one with the same property, as shown below. Therefore $\sqrt{2}$ cannot be rational.



If this is an isosceles right triangle with integer sides,



then there is a smaller one with the same property.

Figure 1

Construction. A circular arc with center at the uppermost vertex and radius equal to the vertical leg of the triangle intersects the hypotenuse at a point, from which a perpendicular to the hypotenuse is drawn to the horizontal leg. Each line segment in the diagram has integer length, and the three segments with double tick marks have equal lengths. (Two of them are tangents to the circle from the same point.) Therefore the smaller isosceles right triangle with hypotenuse on the horizontal base also has integer sides.