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COMPUTING ZEROS OF ANALYTIC MAPPINGS: A LOGARITHMIC RESIDUE APPROACH *

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Abstract.

Let D be a polydisk in $\mathbb{C}^n$ and $f : \overline{D} \to \mathbb{C}^n$ a mapping that is analytic in $\overline{D}$ and has no zeros on the boundary of D. Then f has only a finite number of zeros in D and these zeros are all isolated. We consider the problem of computing these zeros. A multidimensional generalization of the classical logarithmic residue formula from the theory of functions of one complex variable will be our means of obtaining information about the location of these zeros. This integral formula involves the integral of a differential form, which we will transform into a sum of n Riemann integrals of dimension $2n - 1$. We will show how the zeros and their multiplicities can be computed from these integrals by solving a generalized eigenvalue problem that has Hankel structure, and n Vandermonde systems. Numerical examples are included.

AMS subject classification: Primary 65H10; Secondary 32H02.

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1 Introduction.

Consider a polydisk D in $\mathbb{C}^n$ (i.e., D is the Cartesian product of n disks in $\mathbb{C}$) and let $f = (f_1, \ldots, f_n) : \overline{D} \to \mathbb{C}^n$ be a mapping that is analytic in $\overline{D}$ and has no zeros on the boundary of D. The latter implies that f has only a finite number of zeros in D and that these zeros are all isolated [1, Theorem 2.4]. We consider the problem of computing these zeros.

Let $E_f(D)$ denote the set of zeros of f that lie in D and let $\mu_a(f)$ denote the multiplicity of a zero $a \in E_f(D)$.

In case $n = 1$, the classical logarithmic residue formula from the theory of functions of one complex variable tells us that

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$$\frac{1}{2\pi i}\int_{\partial D}\varphi(z)\frac{f'(z)}{f(z)}\,dz = \sum_{a\in E_f(D)}\mu_a(f)\,\varphi(a) \tag{1.1}$$

if $\varphi : \overline{D} \to \mathbb{C}$ is analytic in D and continuous in $\overline{D}$. This integral formula enables one to obtain information about the zeros of f that lie in D. For example, by evaluating the integral in (1.1) with $\varphi(z) \equiv 1$ numerically, one obtains the total number of zeros (i.e., the number of zeros where each zero is counted according to its multiplicity) of f that lie in D.

Methods for the determination of zeros of analytic functions that are based on the numerical evaluation of integrals are called *quadrature methods*. A review of such methods is given in [11].

By setting $\varphi(z) \equiv z^p$, $p = 0, 1, 2, \ldots$, and evaluating the integral in (1.1) numerically for sufficiently many values of p, one obtains a system of polynomial equations for the zeros $a \in E_f(D)$ and their respective multiplicities. To get rid of the multiplicities, Delves and Lyness [5] considered the system

$$Z_1^p + \cdots + Z_N^p = \frac{1}{2\pi i}\int_{\partial D} z^p \frac{f'(z)}{f(z)}\,dz, \qquad p = 1, \ldots, N, \tag{1.2}$$

where N denotes the total number of zeros of f that lie in D and $Z_1, \ldots, Z_N$ denote these zeros. In the sequence $Z_1, \ldots, Z_N$ each zero is repeated according to its multiplicity. Using Newton's identities, they constructed the monic polynomial of degree N whose zeros are given by $Z_1, \ldots, Z_N$. Li [14] revised the method of Delves and Lyness and solved the system of polynomial equations (1.2) directly by using a homotopy continuation method. In our approach the mutually distinct zeros and their respective multiplicities are considered separately. This leads to problems of structured numerical linear algebra, in particular a generalized eigenvalue problem that has Hankel structure.

A multidimensional generalization of (1.1) is available in the theory of functions of several complex variables [1, Theorem 3.1]. It involves the integral of a differential form. To prepare for the numerical evaluation of this integral, we transform it into a Riemann integral, or rather, a sum of n Riemann integrals. This result is formulated in Theorem 2.1 and looks as follows:

$$I(\varphi) := \sum_{k=1}^{n} I_k(\varphi) = \sum_{a\in E_f(D)}\mu_a(f)\,\varphi(a)$$

if $\varphi : \overline{D} \to \mathbb{C}$ is analytic in D and continuous in $\overline{D}$, and where $I_1(\varphi), \ldots, I_n(\varphi)$ are certain Riemann integrals over the unit cube in $\mathbb{R}^{2n-1}$.

The proof of Theorem 2.1 is formulated in the language of differential forms. This cannot be avoided. Readers who feel less at ease with differential forms may consult [4, 8] or [24] for an introduction, and [16] for a thorough exposition of analysis on complex manifolds.

The efficient numerical evaluation of the integrals in Theorem 2.1 requires further study. They are similar to the Kronecker–Picard integrals that appear in

topological degree based methods for computing solutions to twice continuously-differentiable systems of real equations [12, 17, 18, 22, 23].

In this paper however we will assume that we are able to evaluate the functional $I(\varphi)$ for every function φ that satisfies the hypotheses of Theorem 2.1. In particular, we will suppose that the total number of zeros of f that lie in D can be computed. Our unknowns are the number of mutually distinct zeros of f that lie in D, these zeros themselves, and their respective multiplicities. In Section 3 we will show how specific choices of φ enable us to calculate our unknowns by solving problems of structured numerical linear algebra (involving Hankel and Vandermonde matrices). We conclude with numerical examples in Section 4.

Algebraic mappings are of course a special case of analytic mappings. Systems of polynomial equations have received considerable interest in recent years. Several classes of methods have been developed for their solution: homotopy continuation, Groebner bases, sparse resultants and interval methods (see, for example, [6, 7, 15, 20, 21]). We will not compare our approach with these methods because they are specifically for systems of polynomial equations, whereas we consider systems of arbitrary analytic equations, a problem that has received much less attention in the literature.

2 A multidimensional logarithmic residue formula.

Let J_f denote the Jacobian matrix of f and let $J_{[k]}$ be the Jacobian matrix of f with the kth column replaced with $\begin{bmatrix} f_1 & \cdots & f_n \end{bmatrix}^T$:

$$J_{[k]} := \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_{k-1}} & f_1 & \frac{\partial f_1}{\partial z_{k+1}} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_{k-1}} & f_n & \frac{\partial f_n}{\partial z_{k+1}} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}, \qquad k = 1, \ldots, n.$$

Suppose also that the polydisk D is given by

$$D = D_1 \times \cdots \times D_n$$

where

$$D_k = \{ z \in \mathbb{C} : |z - C_k| < R_k \}, \qquad k = 1, \ldots, n,$$

with $C_1, \ldots, C_n \in \mathbb{C}$ and $R_1, \ldots, R_n > 0$.

THEOREM 2.1. *Let $\varphi : \overline{D} \to \mathbb{C}$ be analytic in D and continuous in $\overline{D}$. Define $I_k(\varphi)$ for $k = 1, \ldots, n$ as the integral*

$$I_k(\varphi) := \rho_k \int_{[0,1]^{2n-1}} \frac{\varphi(z_1, \ldots, z_n) \det J_f(z_1, \ldots, z_n) \overline{\det J_{[k]}(z_1, \ldots, z_n)}}{\left(|f_1(z_1, \ldots, z_n)|^2 + \cdots + |f_n(z_1, \ldots, z_n)|^2 \right)^n} \times e^{2\pi i \theta_k} r_1 \cdots r_{k-1} r_{k+1} \cdots r_n \, dr_1 \cdots dr_{k-1} \, dr_{k+1} \cdots dr_n \, d\theta_1 \cdots d\theta_n$$

with

$$\rho_k = \rho(n, R_1, \ldots, R_n; k) := 2^{n-1}(n-1)! \, R_1^2 \cdots R_{k-1}^2 \, R_k \, R_{k+1}^2 \cdots R_n^2$$

and where

$$z_k = z_k(\theta_k) = C_k + R_k\, e^{2\pi i\,\theta_k}, \qquad 0 \le \theta_k \le 1,$$

and

$$z_l = z_l(r_l, \theta_l) = C_l + r_l\, R_l\, e^{2\pi i\,\theta_l}, \qquad 0 \le r_l, \theta_l \le 1,$$

for $l \in \{1, \dots, n\} \setminus \{k\}$. *Then*

$$I(\varphi) := \sum_{k=1}^{n} I_k(\varphi) = \sum_{a \in E_f(D)} \mu_a(f)\, \varphi(a). \tag{2.1}$$

Theorem 2.1 is a corollary of the following theorem.

THEOREM 2.2. *Let D be a polydisk in $\mathbb{C}^n$ and let $f = (f_1, \dots, f_n) : \overline{D} \to \mathbb{C}^n$ be a mapping that is analytic in $\overline{D}$ and has no zeros on the boundary of D. Let $\varphi : \overline{D} \to \mathbb{C}$ be analytic in D and continuous in $\overline{D}$. Define the differential form $\omega(f, \overline{f})$ as*

$$\omega(f, \overline{f}) := \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|f|^{2n}} \sum_{k=1}^{n} (-1)^{k-1}\, \overline{f}_k\, d\overline{f}_{[k]} \wedge df.$$

Then

$$\int_{\partial D} \varphi\, \omega(f, \overline{f}) = \sum_{a \in E_f(D)} \mu_a(f)\, \varphi(a).$$

PROOF. Yuzhakov and Roos [1, Theorem 3.1] proved this result for arbitrary bounded domains in $\mathbb{C}^n$ with a piecewise smooth boundary. □

The notations used in the formulation of the previous theorem,

$$\begin{aligned} \overline{f} &= (\overline{f_1}, \dots, \overline{f_n}), \\ |f| &= \sqrt{|f_1|^2 + \cdots + |f_n|^2}, \\ df &= df_1 \wedge \cdots \wedge df_n, \\ df_{[k]} &= df_1 \wedge \cdots \wedge \widetilde{df_k} \wedge \cdots \wedge df_n, \quad k = 1, \dots, n, \end{aligned}$$

are classical. (The tilde over the form df_k means that this form is omitted and does not appear in the product.)

To get rid of the differential forms in Theorem 2.2, we proceed as follows. Define $\eta(f)$ as

$$\eta(f) := \sum_{k=1}^{n} (-1)^{k-1} f_k\, df_1 \wedge \cdots \wedge \widetilde{df_k} \wedge \cdots \wedge df_n.$$

This form is sometimes called the *Leray form* [13]. Then

$$\omega(f, \overline{f}) = \frac{(n-1)!}{(2\pi i)^n} \frac{1}{|f|^{2n}}\, \eta(\overline{f}) \wedge df.$$

If $f = f(z_1, \dots, z_n)$, then $df = \det J_f(z_1, \dots, z_n)\, dz_1 \wedge \cdots \wedge dz_n$. The following lemma shows what happens with the Leray form in this case.

LEMMA 2.3. *If* $f = f(z_1, \dots, z_n)$, *then*

$$\eta(f) = \sum_{k=1}^{n} (-1)^{k-1} \det J_{[k]}(z_1, \dots, z_n)\, dz_1 \wedge \cdots \wedge \widetilde{dz_k} \wedge \cdots \wedge dz_n.$$

PROOF. Let j be an integer between 1 and n, $j \in \{1, \dots, n\}$. Then

$$\sum_{k=1}^{n} (-1)^{k-1} \det J_{[k]}\, (dz_1 \wedge \cdots \wedge \widetilde{dz_k} \wedge \cdots \wedge dz_n) \left(\frac{\partial}{\partial z_1}, \dots, \widetilde{\frac{\partial}{\partial z_j}}, \dots, \frac{\partial}{\partial z_n} \right)$$
$$= (-1)^{j-1} \det J_{[j]}.$$

By expanding $\det J_{[j]}$ along the jth column, we obtain

$$(-1)^{j-1} \det J_{[j]}$$
$$= (-1)^{j-1} \sum_{k=1}^{n} (-1)^{k+j} f_k \det \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_{j-1}} & \frac{\partial f_1}{\partial z_{j+1}} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_{k-1}}{\partial z_1} & \cdots & \frac{\partial f_{k-1}}{\partial z_{j-1}} & \frac{\partial f_{k-1}}{\partial z_{j+1}} & \cdots & \frac{\partial f_{k-1}}{\partial z_n} \\ \frac{\partial f_{k+1}}{\partial z_1} & \cdots & \frac{\partial f_{k+1}}{\partial z_{j-1}} & \frac{\partial f_{k+1}}{\partial z_{j+1}} & \cdots & \frac{\partial f_{k+1}}{\partial z_n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_{j-1}} & \frac{\partial f_n}{\partial z_{j+1}} & \cdots & \frac{\partial f_n}{\partial z_n} \end{bmatrix}.$$

The determinant in the right hand side is by definition equal to

$$(df_1 \wedge \cdots \wedge \widetilde{df_k} \wedge \cdots \wedge df_n)\left(\frac{\partial}{\partial z_1}, \dots, \widetilde{\frac{\partial}{\partial z_j}}, \dots, \frac{\partial}{\partial z_n}\right).$$

□

It follows that

$$\omega(f, \overline{f}) = \sum_{k=1}^{n} \omega_k(f, \overline{f})$$

where

$$\omega_k(f, \overline{f}) := (-1)^{k-1} \frac{(n-1)!}{(2\pi i)^n} \frac{\det J_f \overline{\det J_{[k]}}}{|f|^{2n}}\, d\overline{z}_{[k]} \wedge dz, \qquad k = 1, \dots, n.$$

The boundary of D is given by

$$\partial D = \partial D_{[1]} \cup \cdots \cup \partial D_{[n]}$$

where

$$\partial D_{[k]} := D_1 \times \cdots \times D_{k-1} \times \partial D_k \times D_{k+1} \times \cdots \times D_n, \qquad k = 1, \dots, n,$$

and thus

$$\int_{\partial D} \varphi\,\omega(f,\overline{f}) = \sum_{k=1}^{n} \int_{\partial D_{[k]}} \varphi\,\omega_k(f,\overline{f}).$$

Next we introduce polar coordinates. Let $C \in \mathbb{C}$ and $R > 0$. If $z = z(\theta) = C + Re^{2\pi i\theta}$, then $dz = 2\pi i\, Re^{2\pi i\theta}\, d\theta$. If $z = z(r,\theta) = C + rRe^{2\pi i\theta}$, then $d\overline{z} \wedge dz = 2(2\pi i) r R^2\, dr \wedge d\theta$. Thus

$$\int_{\partial D_{[k]}} \varphi\,\omega_k(f,\overline{f})$$
$$= (-1)^{k-1}\,\rho_k \int_{\partial D_{[k]}} \frac{\varphi\,\det J_f\,\overline{\det J_{[k]}}}{(|f_1|^2 + \cdots + |f_n|^2)^n} \Big[\prod_{l=1,\, l\neq k}^{n} r_l\Big]\, e^{2\pi i\theta_k}\, dr_{[k]} \wedge d\theta$$

for $k = 1, \ldots, n$, where

$$\rho_k = \rho(n, R_1, \ldots, R_n; k) := 2^{n-1}(n-1)!\, R_1^2 \cdots R_{k-1}^2\, R_k\, R_{k+1}^2 \cdots R_n^2$$

and in the integral on the right hand side

$$z_k = z_k(\theta_k) = C_k + R_k\, e^{2\pi i\theta_k}$$

and

$$z_l = z_l(r_l, \theta_l) = C_l + r_l\, R_l\, e^{2\pi i\theta_l}$$

for $l \in \{1, \ldots, n\} \setminus \{k\}$. In [1] it is assumed that $\mathbb{C}^n$ has the orientation determined by the form $dr_1 \wedge \cdots \wedge dr_n \wedge d\theta_1 \wedge \cdots \wedge d\theta_n$ and that the boundary of D is assigned the orientation induced by that of D. This implies that $\partial D_{[k]}$ has the orientation determined by the form $(-)^{k-1}\, dr_{[k]} \wedge d\theta$. Therefore

$$\int_{\partial D_{[k]}} \varphi\,\omega_k(f,\overline{f}) = \rho_k \int_{[0,1]^{2n-1}} \frac{\varphi(z_1,\ldots,z_n)\,\det J_f(z_1,\ldots,z_n)\,\overline{\det J_{[k]}(z_1,\ldots,z_n)}}{\big(|f_1(z_1,\ldots,z_n)|^2 + \cdots + |f_n(z_1,\ldots,z_n)|^2\big)^n}$$
$$\times\, e^{2\pi i\theta_k}\, r_1 \cdots r_{k-1}\, r_{k+1} \cdots r_n\, dr_1 \cdots dr_{k-1}\, dr_{k+1} \cdots dr_n\, d\theta_1 \cdots d\theta_n$$

for $k = 1, \ldots, n$, where

$$z_k = z_k(\theta_k) = C_k + R_k\, e^{2\pi i\theta_k}, \qquad 0 \le \theta_k \le 1,$$

and

$$z_l = z_l(r_l, \theta_l) = C_l + r_l\, R_l\, e^{2\pi i\theta_l}, \qquad 0 \le r_l, \theta_l \le, 1$$

for $l \in \{1, \ldots, n\} \setminus \{k\}$. This proves Theorem 2.1.

3 The algorithm.

We will now show how the zeros and their respective multiplicities can be computed by solving a generalized eigenvalue problem that has Hankel structure, and n Vandermonde systems.

First we introduce some notation. Let N denote the total number of zeros of f that lie in D. As explained in Section 1 we will assume that the value of N can be computed numerically. From now on, we will also suppose that $N > 0$. Let d be the number of mutually distinct zeros of f that lie in D. Let $(z_1^{(1)}, \ldots, z_n^{(1)}), \ldots, (z_1^{(d)}, \ldots, z_n^{(d)})$ denote these zeros, and let $\mu_1, \ldots, \mu_d$ be their respective multiplicities. Without loss of generality we may assume that $z_n^{(p)} \neq z_n^{(q)}$, $p \neq q$. Indeed, if one first applies a random unitary linear transformation to the unknowns $z = (z_1, \ldots, z_n)$, then this condition is satisfied almost surely, i.e., with probability one. Analogous results can be formulated in case $z_k^{(p)} \neq z_k^{(q)}$, $p \neq q$, for some $k \in \{1, \ldots, n-1\}$. We leave this to the reader. What happens if our algorithm is applied in case the nth components of the zeros of f that lie in D are not mutually distinct, is discussed in Remark 3.2 below.

Define s_p for $p = 0, 1, 2, \ldots$ as

$$s_p := I(z_n^p)$$

where $I(\cdot)$ is defined in (2.1). We will assume that the sequence $(s_p)_{p \geq 0}$ can be computed numerically. By Theorem 2.1

$$s_p = \mu_1 [z_n^{(1)}]^p + \cdots + \mu_d [z_n^{(d)}]^p, \qquad p = 0, 1, 2, \ldots . \tag{3.1}$$

In particular, $s_0 = N$. Let H be the infinite Hankel matrix

$$H := \left[s_{k+l} \right]_{k,l=0}^{\infty}$$

and let

$$H_k := \begin{bmatrix} s_0 & s_1 & \cdots & s_{k-1} \\ s_1 & & \iddots & \vdots \\ \vdots & \iddots & & \vdots \\ s_{k-1} & \cdots & \cdots & s_{2k-2} \end{bmatrix}, \qquad k = 1, 2, \ldots,$$

be its $k \times k$ leading principal submatrix. Also, let V_d be the Vandermonde matrix with nodes $z_n^{(1)}, \ldots, z_n^{(d)}$,

$$V_d := \left[[z_n^{(l)}]^{k-1} \right]_{k,l=1}^{d}.$$

THEOREM 3.1. *The matrices H_1 and H_d are regular. Also, H_k is singular for all $k > d$.*

PROOF.

- $H_1 = [s_0]$ and $s_0 = N > 0$.
- Equation (3.1) implies that we can factorize the matrix H_d as

$$H_d = V_d D_d V_d^T$$

where $D_d := \operatorname{diag}(\mu_1, \ldots, \mu_d)$. The matrix V_d is regular since $z_n^{(1)}, \ldots, z_n^{(d)}$ are assumed to be mutually distinct, and D_d is regular since $\mu_1, \ldots, \mu_d$ are different from zero. Therefore H_d is regular.

- Let P_d be the monic polynomial of degree d that has $z_n^{(1)}, \ldots, z_n^{(d)}$ as simple zeros,

$$P_d(z) := \prod_{k=1}^{d} (z - z_n^{(k)}) =: z^d + \tau_1 z^{d-1} + \cdots + \tau_d.$$

Then

$$\sum_{k=1}^{d} \mu_k [z_n^{(k)}]^p P_d(z_n^{(k)}) = 0, \qquad p = 0, 1, 2, \ldots,$$

and therefore

$$s_{d+p} + \tau_1 s_{d-1+p} + \cdots + \tau_d s_p = 0, \qquad p = 0, 1, 2, \ldots. \tag{3.2}$$

This shows that the last column of H_{d+1+p} is a linear combination of the d previous columns for all $p = 0, 1, 2, \ldots$. □

COROLLARY 3.2. $d = \operatorname{rank} H = \operatorname{rank} H_N$.

Since the value of N is known, this corollary enables us to calculate the value of d via a singular value decomposition of H_N.

The unknowns $z_n^{(1)}, \ldots, z_n^{(d)}$ can be calculated by solving a generalized eigenvalue problem that has Hankel structure.

THEOREM 3.3. *Define* $H_d^< := [s_{1+k+l}]_{k,l=0}^{d-1}$. *Then* $\det(H_d^< - z_n^{(k)} H_d) = 0$ *for* $k = 1, \ldots, d$.

PROOF. An easy calculation shows that

$$[s_{\alpha+k+l}]_{k,l=0}^{d-1} = V_d D_d^{(\alpha)} V_d^T \tag{3.3}$$

for $\alpha = 0, 1, 2, \ldots$ where $D_d^{(\alpha)} := \operatorname{diag}(\mu_1 [z_n^{(1)}]^\alpha, \ldots, \mu_d [z_n^{(d)}]^\alpha)$. (We will only need the cases $\alpha = 0$ and $\alpha = 1$.) Now let λ be an eigenvalue of the pencil $H_d^< - \lambda H_d$ with corresponding eigenvector x. Then (3.3) implies that

$$H_d^< x = \lambda H_d x \qquad \Leftrightarrow \qquad (V_d D_d^{(1)} V_d^T) x = \lambda (V_d D_d^{(0)} V_d^T) x.$$

Since V_d is regular, it follows that $D_d^{(1)} y = \lambda D_d^{(0)} y$ where $y := V_d^T x$. □

Once $z_n^{(1)}, \ldots, z_n^{(d)}$ have been found, $\mu_1, \ldots, \mu_d$ can be calculated by solving the Vandermonde system

$$V_d \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_d \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{d-1} \end{bmatrix}. \tag{3.4}$$

REMARK 3.1. Theoretically the $N - d$ smallest singular values of H_N are equal to zero. In practice, as by evaluating the corresponding integrals numerically we

can only obtain approximations for the s_p's and because of roundoff errors in the SVD computation, this will not be the case. If the numerical rank d of H_N is difficult to determine, it is safe to consider H_N as a matrix of full rank and to solve an $N \times N$ generalized eigenvalue problem and associated Vandermonde system.

THEOREM 3.4. *For every integer $\alpha \geq d$ the eigenvalues of the pencil*

$$[s_{1+k+l}]_{k,l=0}^{\alpha-1} - \lambda\, [s_{k+l}]_{k,l=0}^{\alpha-1} \tag{3.5}$$

are given by $z_n^{(1)}, \ldots, z_n^{(d)}$ and $\alpha - d$ eigenvalues that may assume arbitrary values.

PROOF. This follows from Theorem 3.3 and by taking into account that $(s_p)_{p\geq 0}$ is a linear recurring sequence (cf. the proof of Theorem 3.1, equation (3.2)). Note that the latter implies that the matrices

$$[s_{1+k+l}]_{k,l=0}^{\alpha-1} \qquad \text{and} \qquad [s_{k+l}]_{k,l=0}^{\alpha-1}$$

have the same null spaces. □

The previous theorem tells us that the generalized eigenvalue problem (3.5) has $\alpha - d$ eigenvalues that may assume arbitrary values. Each of these indeterminate eigenvalues corresponds to two corresponding zeros on the diagonals of the generalized Schur decomposition of the Hankel matrices $[s_{1+k+l}]_{k,l=0}^{\alpha-1}$ and $[s_{k+l}]_{k,l=0}^{\alpha-1}$. When actually calculated, these diagonal entries are different from zero because of roundoff errors. The quotient of two such corresponding diagonal entries is an eigenvalue that is not the nth component of a zero of f. Fortunately, the corresponding Vandermonde system enables us to detect such spurious "nth components". Indeed, the Vandermonde matrix whose nodes are given by the eigenvalues of (3.5) is still regular and therefore the corresponding Vandermonde system has only one solution, which gives the true nth components their correct corresponding multiplicity and the spurious ones "multiplicity" zero. ◊

Once d, $z_n^{(1)}, \ldots, z_n^{(d)}$ and $\mu_1, \ldots, \mu_d$ have been found, the unknowns

$$z_1^{(1)}, \ldots, z_1^{(d)}, \ldots, z_{n-1}^{(1)}, \ldots, z_{n-1}^{(d)}$$

can be obtained as follows. Define $t_{k,p}$ for $k = 1, \ldots, n-1$ and $p = 0, 1, 2, \ldots$ as

$$t_{k,p} := I(z_k\, z_n^p).$$

We will assume that the sequences $(t_{k,p})_{p\geq 0}$ can be computed for $k = 1, \ldots, n-1$. By Theorem 2.1

$$t_{k,p} = \mu_1\, z_k^{(1)}\, [z_n^{(1)}]^p + \cdots + \mu_d\, z_k^{(d)}\, [z_n^{(d)}]^p$$

for $k = 1, \ldots, n-1$ and $p = 0, 1, 2, \ldots$. It follows that $z_k^{(1)}, \ldots, z_k^{(d)}$ can be

calculated from the solution of the Vandermonde system

$$V_d \begin{bmatrix} \mu_1 z_k^{(1)} \\ \mu_2 z_k^{(2)} \\ \vdots \\ \mu_d z_k^{(d)} \end{bmatrix} = \begin{bmatrix} t_{k,0} \\ t_{k,1} \\ \vdots \\ t_{k,d-1} \end{bmatrix}, \qquad k = 1, \ldots, n-1.$$

Problems of numerical linear algebra involving Vandermonde or Hankel matrices truly deserve their reputation of being ill-conditioned [9, 19]. By moving the origin in the z_n-plane to the arithmetic mean of the nth components of the zeros,

$$z_n' := \frac{\mu_1 z_n^{(1)} + \cdots + \mu_d z_n^{(d)}}{\mu_1 + \cdots + \mu_d} = \frac{I(z_n)}{I(1)},$$

ill-conditioning is reduced significantly. Therefore we will use shifted versions of the integrals s_p and $t_{k,p}$, denoted by $\hat{s}_p$ and $\hat{t}_{k,p}$. The results of this section then lead to the following algorithm.

ALGORITHM 3.1.

1. $N \leftarrow I(1)$
2. $z_n' \leftarrow I(z_n)/N$
3. $\hat{s}_0 \leftarrow N; \quad \hat{s}_1 \leftarrow 0; \quad \hat{s}_p \leftarrow I((z_n - z_n')^p)$ *for* $p = 2, \ldots, 2N-1$
4. $d \leftarrow \operatorname{rank} \left[\hat{s}_{k+l}\right]_{k,l=0}^{N-1}$
5. *Calculate the eigenvalues* $\lambda_1, \ldots, \lambda_d$ *of the pencil*

$$\left[\hat{s}_{1+k+l}\right]_{k,l=0}^{d-1} - \lambda \left[\hat{s}_{k+l}\right]_{k,l=0}^{d-1}.$$

6. $z_n^{(k)} \leftarrow \lambda_k + z_n'$ *for* $k = 1, \ldots, d$
7. *Solve the Vandermonde system*

$$\begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_d \\ \vdots & \vdots & & \vdots \\ \lambda_1^{d-1} & \lambda_2^{d-1} & \ldots & \lambda_d^{d-1} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_d \end{bmatrix} = \begin{bmatrix} \hat{s}_0 \\ \hat{s}_1 \\ \vdots \\ \hat{s}_{d-1} \end{bmatrix}.$$

8. $\hat{t}_{k,p} \leftarrow I(z_k(z_n - z_n')^p)$ *for* $k = 1, \ldots, n-1$ *and* $p = 0, \ldots, d-1$
9. *Solve the Vandermonde system*

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_d \\ \vdots & \vdots & & \vdots \\ \lambda_1^{d-1} & \lambda_2^{d-1} & \dots & \lambda_d^{d-1} \end{bmatrix} \begin{bmatrix} x_{1,1} & \dots & x_{n-1,1} \\ x_{1,2} & \dots & x_{n-1,2} \\ \vdots & & \vdots \\ x_{1,d} & \dots & x_{n-1,d} \end{bmatrix} = \begin{bmatrix} \hat{t}_{1,0} & \dots & \hat{t}_{n-1,0} \\ \hat{t}_{1,1} & \dots & \hat{t}_{n-1,1} \\ \vdots & & \vdots \\ \hat{t}_{1,d-1} & \dots & \hat{t}_{n-1,d-1} \end{bmatrix}.$$

$$z_k^{(l)} \leftarrow x_{k,l}/\mu_l \text{ for } k = 1, \dots, n-1 \text{ and } l = 1, \dots, d.$$

In practice, as explained in Remark 3.1, we may dispense with Step 4 and solve an $N \times N$ generalized eigenvalue problem and associated Vandermonde system. The components of the solution of the latter will be rounded to the nearest integers, and eigenvalues that have "multiplicity" zero are thrown away.

REMARK 3.2. What happens if our algorithm is applied in case the nth components of the zeros of f that lie in D are not mutually distinct? In this case only a subset of the zeros will be determined correctly, namely the zeros whose nth component occurs only once. To illustrate what happens to the other zeros, suppose for example that $z_n^{(1)} = z_n^{(2)}$ whereas $z_n^{(3)}, \dots, z_n^{(d)}$ are mutually distinct, and

$$\{ z_n^{(1)} \} \cap \{ z_n^{(3)}, \dots, z_n^{(d)} \} = \emptyset.$$

Then

$$s_p = (\mu_1 + \mu_2)\,[z_n^{(1)}]^p + \mu_3\,[z_n^{(3)}]^p + \cdots + \mu_d\,[z_n^{(d)}]^p$$

for $p = 0, 1, 2, \dots$, and

$$t_{k,p} = (\mu_1\, z_k^{(1)} + \mu_2\, z_k^{(2)})\,[z_n^{(1)}]^p + \mu_3\, z_k^{(3)}\,[z_n^{(3)}]^p + \cdots + \mu_d\, z_k^{(d)}\,[z_n^{(d)}]^p$$

for $k = 1, \dots, n-1$ and $p = 0, 1, 2, \dots$. It follows that Algorithm 3.1 will replace each group of zeros that have the same nth component by a point in $\mathbb{C}^n$ whose nth component is equal to the nth component that is shared by these zeros and whose other components are given by the arithmetic mean of the corresponding components of these zeros.

4 Numerical examples.

In the following example we have chosen $n = 4$ (the number of variables) and $d = 5$ (the number of mutually distinct zeros). The zeros themselves and their respective multiplicities are listed in the following table.

$(z_1^{(k)}, \dots, z_n^{(k)})$	μ_k
(1, i, -2, 1)	1
(-3, 0, 2, 2)	2
(2, $-i$, 5, 3)	1
(1, 0, -1, 4)	2
(0, 1, 3, 5)	1

Then $N = 7$ (the total number of zeros) and $z_n' = (1{\cdot}1+2{\cdot}2+1{\cdot}3+2{\cdot}4+1{\cdot}5)/7 = 3$ (the arithmetic mean of the last components). The shifted moments for the last components are given by $\hat{s}_p = 1 \cdot (1-3)^p + 2 \cdot (2-3)^p + 1 \cdot (3-3)^p + 2 \cdot (4-$

$3)^p + 1 \cdot (5-3)^p$ and thus $\hat{s}_p = (2+2^p)(1+(-1)^p)$ for $p \geq 1$. We have computed the eigenvalues of the pencil

$$\left[\hat{s}_{1+k+l}\right]_{k,l=0}^{N-1} - \lambda \left[\hat{s}_{k+l}\right]_{k,l=0}^{N-1}$$

using LAPACK's routine CGEGV [2]. The calculations have been performed in single precision arithmetic on an IBM SP2. The results are shown in the following table, in which each generalized eigenvalue is represented as a pair (α_k, β_k) of two corresponding diagonal entries in the generalized Schur decomposition of $\left[\hat{s}_{1+k+l}\right]_{k,l=0}^{N-1}$ and $\left[\hat{s}_{k+l}\right]_{k,l=0}^{N-1}$.

α_k	β_k	$z_n^{(k)}$	μ_k
−30316.83	15158.42	1.000000	1.000001
−2322.758	2322.756	1.999999	1.999996
0.003323	542.2963	3.000006	1.000017
950.4792	950.4736	4.000006	1.999985
33612.10	16806.05	5.000000	0.999998
0.000000	0.000233		
0.000000	0.000102		

The number of mutually distinct zeros is clearly equal to five. The generalized eigenvalues α_k/β_k corresponding to the last two pairs (α_k, β_k) represent spurious nth components and are thrown away. Also shown in this table are the computed values $z'_n + \alpha_k/\beta_k$ of the nth components and the corresponding solutions of the Vandermonde system (3.4) for the multiplicities, which has been solved via the algorithm of Gohberg and Koltracht [10]. This algorithm takes full account of the structure of a Vandermonde matrix and is not only faster but also more accurate than general purpose algorithms such as Gaussian elimination with partial pivoting. This Vandermonde solver has arithmetic complexity $\mathcal{O}(d^2)$. It has also been used to solve the Vandermonde system for the first $n-1$ components (Step 9 in Algorithm 3.1). The results are shown in the following table.

$z_1^{(k)}$	$z_2^{(k)}$	$z_3^{(k)}$
$1.000002 - i\,0.000000$	$0.000000 + i\,1.000001$	$-2.000006 - i\,0.000000$
$-2.999997 + i\,0.000000$	$-0.000000 - i\,0.000002$	$2.000013 + i\,0.000001$
$2.000010 - i\,0.000000$	$0.000000 - i\,1.000002$	$4.999987 - i\,0.000002$
$0.999992 + i\,0.000000$	$-0.000001 + i\,0.000002$	$-1.000007 + i\,0.000001$
$-0.000002 + i\,0.000000$	$1.000000 - i\,0.000000$	$3.000006 - i\,0.000000$

The previous example illustrates how the unknown zeros and multiplicities can be computed from the integrals $\hat{s}_p$ and $\hat{t}_{k,p}$ by Algorithm 3.1. The efficient numerical evaluation of these integrals is a problem that remains to be tackled. Nevertheless we would like to present a small example in which these integrals were calculated numerically.

Consider the problem of computing all the zeros of

$$f = f(z_1, z_2) = (\sin z_1 + z_1^2 + e^{z_2} - \cos(2z_2), \cos z_1 + z_2^3 + e^{2z_2} - 2)$$

that lie in the polydisk $D = \{z_1 \in \mathbb{C} : |z_1| < 1\} \times \{z_2 \in \mathbb{C} : |z_2| < 1\}$. In this case we have to integrate over the unit cube in $\mathbb{R}^3$. The integrals were calculated via DCUHRE [3]. This package implements an adaptive algorithm for numerical integration over hyperrectangular regions. First we calculated $\operatorname{Re} I(1)$. We requested an absolute accuracy of 0.1 and obtained that $\operatorname{Re} I(1) \approx 1.998$. Thus $N = 2$. Next we calculated the arithmetic mean of the second components. A crude approximation to this mean is sufficient to reduce ill-conditioning, and therefore we requested a relative accuracy of only 0.1. Finally we calculated all the other integrals needed by Algorithm 3.1. One of the very interesting features of DCUHRE is that it is able to integrate a *vector* of similar integrals over a common integration region. Since a significant part of the computation required for each integrand is the same for all of the integrands, these common calculations need be done only once for each integrand evaluation point. We requested a relative accuracy of 10^{-5}. DCUHRE needed about 10^5 functional evaluations to obtain this accuracy. With these approximations for the integrals $\hat{s}_p$ and $\hat{t}_{k,p}$ as input, Algorithm 3.1 obtained that f has two zeros in D ($d = 2$), each of multiplicity one. We refined the approximations for these zeros iteratively via Newton's method. The zeros of f that lie in D are given by $(0,0)$ and $(-0.72011062161456, 0.11033979708375)$. Note that the number of functional evaluations needed to approximate the integrals to the requested accuracy is independent of the number of zeros of f that lie in D.

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